1. Let $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, +\infty\}$ with the usual understanding on its ordering etc. (so $\mathbb{R}^*$ is <u>not</u> a field); in particular we have the notions of upper/lower bounds (u.b./l.b.).
Let $A \subseteq \mathbb{R}^*$. Define

$$\sup A := \text{the smallest upper bound of } A$$
$$\inf A := \text{the greatest lower bound of } A$$

Warning.

$$\sup \emptyset = -\infty \quad (\because \text{all elements of } \mathbb{R}^* \text{ are u.b. of } \emptyset)$$
$$\inf \emptyset = +\infty$$

($\therefore$ we usually would start with assuming that our A is nonempty, ~~say~~ $a_0 \in A$ so $\inf A \leq a \leq \sup A$)

Show that, $\forall\, \emptyset \neq A \subseteq B \subseteq \mathbb{R}^*$,

1) $\sup A \leq \sup B$
2) $\inf A \geq \inf B$
3) $\sup(A+B) \leq \sup A + \sup B$
4) $\inf(A+B) \geq \inf A + \inf B$
5) $\sup(-A) = -\inf A$,

where $-A := \{-a : a \in A\}$

$$A+B := \{a+b : a \in A,\ b \in B\}$$

2. Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of sets and $B_1 = A_1$ & $B_n := A_n \setminus \left(\bigcup_{i<n} A_i\right)$ $\forall n > 1$
Show (by the Well-order principle) that $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n$.

3. Let $f : A \to \mathbb{R}$ (A, for simplicity, an interval) and $x_0 \in A$. We say that f is lower semi-continuous (lsc) at x_0 if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s.t.

$$f(x_0) - \varepsilon < f(x) \quad \forall\, x \in A \cap V_\delta(x_0)$$

Show that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii), where

(i) f is lsc at x_0

(ii) $f(x_0) \leq \sup\limits_{\delta > 0} \inf\limits_{u \in A \cap V_\delta(x_0)} f(u)$ (to be denoted by $\liminf\limits_{u \to x_0} f(u)$ or $\underline{f}(x_0)$)

(iii) $f(x_0) \leq \sup\limits_{\delta > 0} \inf\limits_{u \in (A \setminus \{x_0\}) \cap V_\delta(x_0)} f(u)$

Note. In general, the RHS of (ii), (iii) are not the same, even though RHS of (iii) may also be denoted by the same notation.

4.* Let $(X, \mathcal{A}, \mu)$ be a "measure space": X is a set, $\mathcal{A}$ an σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, +\infty]$ a measure. Show that

1) If $A \subseteq B$ and $\mu(A) < +\infty$ then $m(B \setminus A) = m(B) - m(A)$.

2) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ with $A_n \subseteq A_{n+1}$ $\forall\, n$. Show that $m(A_n) \leq m(A_{n+1})$ $\forall\, n$ and $m\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \lim\limits_n m(A_n)$.

3) Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathcal{A}$ with $A_n \supseteq A_{n+1}$ $\forall\, n$. Show that $\lim\limits_n \mu(A_n) = \mu\left(\bigcap\limits_{n \in \mathbb{N}} A_n\right)$, provided that $\mu(A_N) < +\infty$ for some $N \in \mathbb{N}$.

5*. In $\mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$, show the "Generalized" Monotone Convergence Theorem for sequences of extended-real numbers: If (a_n) is a monotone seq of extended-real numbers then it converges to a limit in $\mathbb{R}^*$. Show further that the

$$\limsup x_n := \bigwedge_{k=1}^{\infty} \bigvee_{n \geq k} x_n \left(:= \inf_{k \in \mathbb{N}} \left(\sup_{n \geq k} x_n \right) \right)$$

$$\liminf x_n := \bigvee_{k=1}^{\infty} \bigwedge_{n \geq k} x_n$$

exist in $\mathbb{R}^*$, and that

$$\liminf x_n = \limsup x_n \text{ iff } \lim_n x_n \text{ exists}$$

(and all the three are same then).